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Y.-K. LAU, G.-S. LÜ & J. WU

Abstract. In this paper, we investigate the ℓth power sum of Hecke eigenvalues of classical holomorphic cusp forms for $3 \leqslant \ell \leqslant 8$ and improve the related results of Lü [15, 16, 17]. We also establish Ω-estimates for $2 \leqslant \ell \leqslant 6$ and affirm a conjecture of Ivić [7, (7.23)].

1. Introduction

The set of primitive holomorphic forms of even integral weight $k \geqslant 2$ for the full modular group $SL(2, \mathbb{Z})$, denoted by H_k^*, consists of the common eigenfunctions f of all Hecke operators T_n, whose Fourier series expansions at the cusp ∞ are of form

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} \mathrm{e}^{2\pi \mathrm{i} nz} \qquad (\Im m z > 0), \tag{1.1}$$

and the coefficients $\lambda_f(n)$ are (Hecke) eigenvalues of T_n. As a function of n, $\lambda_f(n)$ is real-valued and multiplicative. Furthermore, it was shown by Deligne that for every prime p there is a (complex) number $\alpha_f(p)$ such that

$$|\alpha_f(p)| = 1 \quad \text{and} \quad \lambda_f(p^\nu) = \alpha_f(p)^\nu + \alpha_f(p)^{\nu-2} + \cdots + \alpha_f(p)^{-\nu} \tag{1.2}$$

for all integers $\nu \geqslant 1$. This follows the Deligne inequality

$$|\lambda_f(n)| \leqslant d(n) \tag{1.3}$$

for all integers $n \geqslant 1$, where $d(n)$ is the divisor function.

In this paper we consider, for $f \in \mathrm{H}_k^*$, the asymptotic comportment of the ℓth power sum $S_\ell(f; x)$ of the Hecke eigenvalues, defined as

$$S_\ell(f; x) := \sum_{n \leqslant x} \lambda_f(n)^\ell$$

where $\ell \in \mathbb{N}$ and $x \geqslant 1$.

1.1. ***O*-results on $S_\ell(f; x)$.** This problem received attention of many authors (see [26] for a detailed historical description). For $\ell = 1$, the best result to date is (given by [26, Theorem 3]):

$$S_1(f; x) \ll_f x^{1/3} (\log x)^{\rho_{1/2}^+}$$

where $\rho_{1/2}^+ := \frac{102+7\sqrt{21}}{210}\big(\frac{6-\sqrt{21}}{5}\big)^{1/2} + \frac{102-7\sqrt{21}}{210}\big(\frac{6+\sqrt{21}}{5}\big)^{1/2} - \frac{33}{35} = -0.118\dots$. When the Sato-Tate conjecture holds, $\rho_{1/2}^+$ can be replaced by $\theta_{1/2} := \frac{8}{3\pi} - 1 = -0.151\dots$.

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The case $\ell = 2$ is the well known result obtained independently by Rankin [22] and Selberg [24]. Their powerful method, now entitled the Rankin-Selberg method, gives

$$S_2(f;x) = C_f x + O_f(x^{3/5}),$$

where C_f is a positive constant depending on f. † A key point of their method is the analytic properties of the Rankin-Selberg L-function

$$L(s, f \times f) := \zeta(2s) \sum_{n \geqslant 1} \lambda_f(n)^2 n^{-s}.$$

As usual, $\zeta(s)$ denotes the Riemann zeta-function.

The study of $S_\ell(f;x)$ for other ℓ requires akin auxiliary tools. Associated to each $f \in \mathrm{H}_k^*$, we have the symmetric mth power L-function ($m \in \mathbb{N}$) defined by

$$L(s, \mathrm{sym}^m f) := \prod_p \prod_{0 \leqslant j \leqslant m} \big(1 - \alpha_f(p)^{m-2j} p^{-s}\big)^{-1} \tag{1.4}$$

for $\sigma > 1$, where and in the sequel we write $s = \sigma + \mathrm{i}\tau$. With (1.2), one has

$$L(s, f \times f) = \zeta(s) L(s, \mathrm{sym}^2 f) \tag{1.5}$$

for $\Re e s > 1$. Using Moreno & Shahidi's work [19] on $L(s, \mathrm{sym}^m f)$ for $m = 2, 3, 4$, Fomenko [1, Theorems 1 and 4] established the following estimates:

$$S_3(f;x) \ll_{f,\varepsilon} x^{5/6+\varepsilon}, \qquad S_4(f;x) = D_f x \log x + F_f x + O_{f,\varepsilon}(x^{9/10+\varepsilon}),$$

where D_f and F_f are constants depending on f and ε is an arbitrarily small positive number. Recently Lü improved Fomenko's results and investigated the higher moments:

$$S_\ell(f;x) = x P_\ell(\log x) + O_{f,\varepsilon}\big(x^{\theta_\ell + \varepsilon}\big) \quad (3 \leqslant \ell \leqslant 8), \tag{1.6}$$

where $P_\ell(t) \equiv 0$ for $\ell = 3, 5, 7$, $P_4(t), P_6(t), P_8(t)$ are polynomials of degree $1, 4, 13$ respectively and

$$\begin{array}{lll} \theta_3 = \frac{3}{4} = 0.75, & \theta_5 = \frac{15}{16} = 0.9375, & \theta_7 = \frac{63}{64} = 0.984375. \\ \theta_4 = \frac{7}{8} = 0.875, & \theta_6 = \frac{31}{32} = 0.96875, & \theta_8 = \frac{127}{128} = 0.99218\ldots, \end{array} \tag{1.7}$$

See [15, Theorem 1.1 and 1.4], [16, Theorems 1.1 and 1.2] and [17, Theorems 1.1 and 1.2]. A key ingredient of the proof is the properties of $L(s, \mathrm{sym}^m f)$ ($m = 5, 6, 7, 8$) in the excellent work of Kim & Shahidi [11].

Our first aim is to refine the exponents θ_ℓ.

Theorem 1. *Under the previous notation, we have*

$$\begin{array}{lll} \theta_3 = \frac{7}{10} = 0.7, & \theta_5 = \frac{40}{43} = 0.9302\ldots, & \theta_7 = \frac{176}{179} = 0.9832\ldots, \\ \theta_4 = \frac{151}{175} = 0.8628\ldots, & \theta_6 = \frac{175}{181} = 0.9668\ldots, & \theta_8 = \frac{2933}{2957} = 0.9918\ldots \end{array}$$

For $f \in \mathrm{H}_k^*$, $\ell \in \mathbb{N}$ and $\Re e s > 1$, let us define

$$F_\ell(s) := \sum_{n \geqslant 1} \lambda_f(n)^\ell n^{-s}. \tag{1.8}$$

It is known that $F_\ell(s)$ factorizes into

$$F_\ell(s) = G_\ell(s) H_\ell(s) \tag{1.9}$$

†The exponent 3/5 in the error term remains the best since its birth.

where $G_\ell(s)$ is product of the Riemann ζ-function and $L(s, \mathrm{sym}^m f)$ with $m \leqslant \ell$, and $H_\ell(s)$ is a Dirichlet series absolutely convergent in $\Re e\, s > \frac{1}{2}$ (see [26, Lemma 2.4], for example). Since the automorphy of $L(s, \mathrm{sym}^m f)$ is available only when $m \leqslant 4$, the cases $5 \leqslant \ell \leqslant 8$ cannot be treated directly. The basic idea of Lü to overcome this difficulty is the use of the Rankin-Selberg L-functions attached to $\mathrm{sym}^m f$ and $\mathrm{sym}^n f$,

$$L(s, \mathrm{sym}^m f \times \mathrm{sym}^n f) := \prod_p \prod_{0 \leqslant j \leqslant m} \prod_{0 \leqslant \ell \leqslant n} \big(1 - \alpha_f(p)^{m-2j} \alpha_f(p)^{n-2\ell} p^{-s}\big)^{-1}$$

for $\Re e s > 1$. See [15, (3.1)], [16, Lemmas 2.1 and 2.2], also [13, Lemma 7.2]. When $\ell \leqslant 8$, the theory for general Rankin-Selberg L-functions guarantees that $G_\ell(s)$ is a *general* L-function in the sense of Perelli [21]. The values of θ_ℓ in (1.7) is obtained with the (individual or averaged) convexity bounds for general L-functions.

The main idea for our improvement is an alternative expression of $G_\ell(s)$ in Lemma 2.1 below, different from [15, 16, 17, 13]; this expression decomposes $G_\ell(s)$ into a product of L-functions, *general* and (more importantly) of lower degree ($\leqslant 3$). Hence we can take advantage of their (individual or averaged) subconvexity bounds (see Lemmas 2.3, 2.4 and 2.5 below). Our sharpening relies on these delicate results, and the method also leads to improve [15, Theorems 1.2 and 1.3] and [17, Theorems 1.3, 1.4 and 1.5].

1.2. **Ω-results on $S_\ell(f; x)$.** To realize the quality of O-results on $S_\ell(f; x)$ one may evaluate Ω-estimates. The case of $\ell = 1$ was considered by various authors. Currently the best result is due to Hafner & Ivić [4, Theorem 2]:

$$S_1(f; x) = \Omega_\pm\bigg(x^{1/4} \exp\bigg\{\frac{D(\log\log x)^{1/4}}{(\log\log\log x)^{3/4}}\bigg\}\bigg).$$

For even ℓ, we denote by $\Delta_\ell(f; x)$ the error term in (1.6). Ivić [7, (7.23)] conjectured

$$\Delta_2(f; x) = \Omega(x^{3/8}). \tag{1.10}$$

Our second aim is to establish some Ω-results, which in particular affirms (1.10).

Theorem 2. *Under the previous notation, we have*

$$S_\ell(f; x) = \Omega\big(x^{(1-2^{-\ell})/2}\big) \qquad (\ell = 3, 5), \tag{1.11}$$

$$\Delta_\ell(f; x) = \Omega\big(x^{(1-2^{-\ell})/2}\big) \qquad (\ell = 2, 4, 6). \tag{1.12}$$

Our principal tool in the proof is Kühleitner & Nowak's general Omega theorem for a class of arithmetic functions [12, Theorem 2]. To implement it, we need a more precise decomposition of the Dirichlet series $H_\ell(s)$ in (1.9) (see Lemma 4.1). For the sake of unconditional results, we are restricted to $\ell \leqslant 6$ because the automorphy of $L(s, \mathrm{sym}^m f)$ is merely available for $m = 1, 2, 3, 4$.

2. Preliminary lemmas

This section is devoted to establish and recall some preliminary results for the proof of Theorem 1.

2.1. **Decomposition of $F_\ell(s)$.** As indicated in the introduction, our starting point is a new decomposition of $F_\ell(s)$.

Lemma 2.1. *Let $f \in \mathrm{H}_k^*$. Then we have*

$$F_\ell(s) = G_\ell(s)H_\ell(s) \tag{2.1}$$

for $\ell = 3, \dots, 8$, where

$$\begin{aligned}
G_3(s) &= L(s,f)^2 L(s,\mathrm{sym}^3 f),\\
G_4(s) &= \zeta(s)^2 L(s,\mathrm{sym}^2 f)^3 L(s,\mathrm{sym}^4 f),\\
G_5(s) &= L(s,f)^5 L(s,\mathrm{sym}^3 f)^3 L(s,\mathrm{sym}^4 f\times f),\\
G_6(s) &= \zeta(s)^5 L(s,\mathrm{sym}^2 f)^8 L(s,\mathrm{sym}^4 f)^4 L(s,\mathrm{sym}^4 f\times \mathrm{sym}^2 f),\\
G_7(s) &= L(s,f)^{13} L(s,\mathrm{sym}^3 f)^8 L(s,\mathrm{sym}^4 f\times f)^5 L(s,\mathrm{sym}^4 f\times \mathrm{sym}^3 f),\\
G_8(s) &= \zeta(s)^{13} L(s,\mathrm{sym}^2 f)^{21} L(s,\mathrm{sym}^4 f)^{13} L(s,\mathrm{sym}^4 f\times \mathrm{sym}^2 f)^6\\
&\qquad \times L(s,\mathrm{sym}^4 f\times \mathrm{sym}^4 f),
\end{aligned}$$

and the function $H_\ell(s)$ admits a Dirichlet series convergent absolutely in $\Re e s > \frac{1}{2}$ and $H_\ell(s) \neq 0$ for $\Re e s = 1$.

Proof. Let $T_n(x)$ (resp. $T_m \times T_n(x)$) be the polynomial which gives the trace of the nth symmetric power of an element (resp. the trace of the Rankin-Selberg convolution of the mth symmetric power and the nth symmetric power) of $SL_2(\mathbb{C})$ whose trace is x. Then $T_m \times T_n(x) = T_m(x)T_n(x)$. Thus the expression (2.1) is a consequence of Lemma 2.2 below with $m = 4$. The absolute convergence of $H_\ell(s)$ for $\Re e s > \frac{1}{2}$ can be easily deduced by the Deligne inequality (1.3). $\square$

Lemma 2.2. *Let $m \in \mathbb{N}$. For $0 \leqslant \ell \leqslant 2m$ and $0 \leqslant j \leqslant 2m+2$, define*

$$a_{\ell,j} := \begin{cases} \binom{\ell}{(\ell-j)/2} - \binom{\ell}{(\ell-j)/2-1} & \text{if } j \equiv \ell \,(\mathrm{mod}\, 2),\\ 0 & \text{otherwise},\end{cases}$$

where $\binom{n}{i}$ is the binomial coefficient with the convention that $\binom{n}{i} = 0$ if $i < 0$. Then

$$x^\ell = \sum_{j=0}^{m-1} (a_{\ell,j} - a_{\ell,2m-j})T_j(x) + \sum_{j=0}^{m} (a_{\ell,m+j} - a_{\ell,m+j+2})T_m(x)T_j(x) \tag{2.2}$$

for $\ell = 0, 1, \dots, 2m$.

Proof. Let $U_n(x)$ be the nth Chebyshev polynomial of the second kind. Then

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}, \qquad T_n(x) = U_n(x/2). \tag{2.3}$$

It is well known that the U_n are orthogonal with respect to the inner product:

$$\langle U_m, U_n\rangle := \frac{2}{\pi}\int_0^\pi U_m(\cos\theta)U_n(\cos\theta)(\sin\theta)^2 \mathrm{d}\theta = \delta_{m,n}, \tag{2.4}$$

where $\delta_{m,n}$ is the Kronecker symbol.

Firstly we establish the following formulas: for $0 \leqslant i, j \leqslant m$,

$$\langle U_m U_i, U_j - U_{2m-j}\rangle = 0, \tag{2.5}$$

$$\langle U_m U_i, U_{m+j} - U_{m+j+2}\rangle = \delta_{i,j}. \tag{2.6}$$

We begin with a simple trigonometric identity (for $0 \leqslant i \leqslant m$)

$$U_m(\cos\theta)U_i(\cos\theta) = \sum_{d=0}^{i} U_{m+i-2d}(\cos\theta), \tag{2.7}$$

which can be verified as follows,

$$\begin{aligned}\sum_{d=0}^{i}\sin((m+i-2d+1)\theta)\sin\theta &= \frac{\cos((m-i)\theta)-\cos((m+i+2)\theta)}{2}\\ &= \sin((m+1)\theta)\sin((i+1)\theta).\end{aligned}$$

Combining this identity with the orthogonality relation (2.4), we deduce that

$$\begin{aligned}\langle U_m U_i, U_j - U_{2m-j}\rangle &= \Big\langle \sum_{d=0}^{i} U_{m+i-2d}, U_j - U_{2m-j}\Big\rangle\\ &= \sum_{d=0}^{i}\langle U_{m+i-2d}, U_j\rangle - \sum_{d=0}^{i}\langle U_{m+i-2d}, U_{2m-j}\rangle\\ &=: A - B.\end{aligned} \tag{2.8}$$

Since $m+i-2d = j \Leftrightarrow m+i-2(i-d) = 2m-j$, A and B takes the same value (which equals 0 or 1) and (2.5) follows from (2.8) immediately.

Similarly, for $0 \leqslant i, j \leqslant m$, we have

$$\langle U_m U_i, U_{m+j} - U_{m+j+2}\rangle = \sum_{d=0}^{i}\langle U_{m+i-2d}, U_{m+j}\rangle - \sum_{d=0}^{i}\langle U_{m+i-2d}, U_{m+j+2}\rangle.$$

Then it is trivial to verify (2.6).

Now we are ready to prove (2.2). Denote by $V_{2m}(x)$ the vector space of all real polynomials of degree $\leqslant 2m$ over $\mathbb{R}$. It is well known that $T_0(x), T_1(x), \ldots, T_{2m}(x)$ constitite a base of $V_{2m}(x)$. In view of the identity

$$T_m(x)T_j(x) = T_{m+j}(x) + T_{m+j-2}(x) + \cdots + T_{m-j}(x) \quad (0 \leqslant j \leqslant m)$$

which is equivalent to (2.7), we easily see that

$$T_0(x), \ldots, T_{m-1}(x), T_m(x)T_0(x), \ldots, T_m(x)T_m(x)$$

constitite a base of $V_{2m}(x)$. Thus for $0 \leqslant \ell \leqslant 2m$, we can write

$$x^\ell = \sum_{j=0}^{m-1} a_{m,\ell}(j)T_j(x) + \sum_{j=0}^{m} b_{m,\ell}(j)T_m(x)T_j(x). \tag{2.9}$$

Therefore it remains to show

$$a_{m,\ell}(j) = a_{\ell,j} - a_{\ell,2m-j} \quad (0 \leqslant j \leqslant m-1), \tag{2.10}$$

$$b_{m,\ell}(j) = a_{\ell,m+j} - a_{\ell,m+j+2} \quad (0 \leqslant j \leqslant m). \tag{2.11}$$

Clearly (2.9) is equivalent to

$$(2x)^\ell = \sum_{j=0}^{m-1} a_{m,\ell}(j)U_j(x) + \sum_{j=0}^{m} b_{m,\ell}(j)U_m(x)U_j(x). \tag{2.12}$$

For $0 \leqslant \ell \leqslant 2m$ and $0 \leqslant j \leqslant 2m+2$, we have

$$\begin{aligned}\langle (2x)^\ell, U_j\rangle &= \frac{2}{\pi}\int_0^\pi (2\cos\theta)^\ell \frac{\sin((j+1)\theta)}{\sin\theta}(\sin\theta)^2\mathrm{d}\theta \\ &= \frac{2^\ell}{\pi}\int_0^\pi (\cos\theta)^\ell\cos(j\theta)\mathrm{d}\theta - \frac{2^\ell}{\pi}\int_0^\pi (\cos\theta)^\ell\cos((j+2)\theta)\mathrm{d}\theta.\end{aligned}$$

In view of the following formula

$$\begin{aligned}\frac{2^\ell}{\pi}\int_0^\pi (\cos\theta)^\ell\cos(j\theta)\mathrm{d}\theta &= \frac{2^\ell}{2\pi\mathrm{i}}\int_{|z|=1}\bigg(\frac{z+z^{-1}}{2}\bigg)^\ell z^{j-1}\mathrm{d}z \\ &= \sum_{d=0}^{\ell}\binom{\ell}{\ell-d}\frac{1}{2\pi\mathrm{i}}\int_{|z|=1} z^{-\ell+j+2d-1}\mathrm{d}z \\ &= \begin{cases}\binom{\ell}{(\ell-j)/2} & \text{if } \ j\equiv \ell \,(\mathrm{mod}\, 2), \\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

we obtain

$$\langle (2x)^\ell, U_j\rangle = a_{\ell,j} \qquad (0 \leqslant \ell \leqslant 2m,\ 0 \leqslant j \leqslant 2m+2). \tag{2.13}$$

For $0 \leqslant j \leqslant m-1$, from (2.13), (2.12), (2.4) and (2.5), we infer that

$$\begin{aligned}a_{\ell,j} - a_{\ell,2m-j} &= \langle (2x)^\ell, U_j - U_{2m-j}\rangle \\ &= \sum_{i=0}^{m-1} a_{m,\ell}(i)\langle U_i, U_j - U_{2m-j}\rangle + \sum_{i=0}^{m} b_{m,\ell}(i)\langle U_mU_i, U_j - U_{2m-j}\rangle \\ &= a_{m,\ell}(j).\end{aligned}$$

Similarly for $0 \leqslant j \leqslant m$, we deduce

$$\begin{aligned}a_{\ell,m+j} - a_{\ell,m+j+2} &= \sum_{i=0}^{m-1} a_{m,\ell}(i)\langle U_i, U_{m+j} - U_{m+j+2}\rangle \\ &\quad + \sum_{i=0}^{m} b_{m,\ell}(i)\langle U_mU_i, U_{m+j} - U_{m+j+2}\rangle \\ &= b_{m,\ell}(j)\end{aligned}$$

by (2.13), (2.12), (2.4) and (2.6) again. This proves (2.10) and (2.11). □

2.2. **Mean values and subconvexity bounds.**

Lemma 2.3. *For any $\varepsilon > 0$, we have*

$$\int_0^T \big|\zeta(\tfrac{5}{7}+\mathrm{i}\tau)\big|^{12}\mathrm{d}\tau \ll_\varepsilon T^{1+\varepsilon} \tag{2.14}$$

uniformly for $T \geqslant 1$, *and*

$$\zeta(\sigma + i\tau) \ll_\varepsilon (|\tau| + 1)^{\max\{(1/3)(1-\sigma),\, 0\}+\varepsilon} \tag{2.15}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 2$ *and* $|\tau| \geqslant 1$.

These are Theorem 8.4 and (8.87) in [5] and Theorem II.3.6 in [25].

Lemma 2.4. *Let* $f \in \mathrm{H}_k^*$ *and* $\varepsilon > 0$. *Then we have*

$$\int_0^T \big|L(\tfrac{5}{8} + \mathrm{i}\tau, f)\big|^4 \mathrm{d}\tau \ll_\varepsilon T^{1+\varepsilon} \tag{2.16}$$

uniformly for $T \geqslant 1$, *and*

$$L(\sigma + \mathrm{i}\tau, f) \ll_{f,\varepsilon} (|\tau| + 1)^{\max\{(2/3)(1-\sigma),\, 0\}+\varepsilon} \tag{2.17}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 2$ *and* $|\tau| \geqslant 1$.

These are [6, Theorem 2, (1.8)] and [3, Corollary], respectively.

Lemma 2.5. ([14, Corollary 1.2]) *Let* $f \in \mathrm{H}_k^*$ *and* $\varepsilon > 0$. *Then we have*

$$L(\sigma + \mathrm{i}\tau, \mathrm{sym}^2 f) \ll_{f,\varepsilon} (|\tau| + 1)^{\max\{(11/8)(1-\sigma),\, 0\}+\varepsilon} \tag{2.18}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 2$ *and* $|\tau| \geqslant 1$.

2.3. **Mean values and convexity bound for higher rank L-functions.** For our purpose we need an immediate consequence of Perelli's mean value theorem and convexity bound for the *general* L-function in [21].

For $\mathbf{d} := \{d_1, \dots, d_J\}$, $\mathbf{m} := \{m_1, \dots, m_J\}$, $\mathbf{n} := \{n_1, \dots, n_J\}$ with $d_j \in \mathbb{N}$, $1 \leqslant m_j \leqslant 4$ and $0 \leqslant n_j \leqslant m_j$, define

$$\mathfrak{L}_{\mathbf{m},\mathbf{n}}^{\mathbf{d}}(s) := \prod_{j=1}^{J} L(s, \mathrm{sym}^{m_j} f \times \mathrm{sym}^{n_j} f)^{d_j}, \tag{2.19}$$

where we take the convention that

$$\begin{cases} L(s, \mathrm{sym}^0 f) = \zeta(s), \\ L(s, \mathrm{sym}^1 f) = L(s, f), \\ L(s, \mathrm{sym}^m f \times \mathrm{sym}^0 f) = L(s, \mathrm{sym}^m f). \end{cases}$$

The works of Hecke (see [8]), Gelbart & Jacquet [2], Kim [9] and Kim & Shahidi [10, 11] show that $L(s, \mathrm{sym}^m f)$ ($1 \leqslant m \leqslant 4$) is a *general* L-function, and so are $L(s, \mathrm{sym}^m f \times \mathrm{sym}^n f)$ for $m, n \leqslant 4$ by [23]. Plainly $\mathfrak{L}_{\mathbf{m},\mathbf{n}}^{\mathbf{d}}(s)$ is also a *general* L-function whose parameters $\alpha_j = \frac{1}{2}$, $\beta_j \geqslant 0$ for all j and

$$M = N = d_1(m_1 + 1)(n_1 + 1) + \cdots + d_J(m_J + 1)(n_J + 1)$$

with the notation as in [21]. Thus

$$A := \tfrac{1}{2}\{d_1(m_1 + 1)(n_1 + 1) + \cdots + d_J(m_J + 1)(n_J + 1)\}, \quad B \geqslant 0$$

and

$$H := 1 + \Re e(B/A) - (N - 1)/(2A) \geqslant 1/N > 0.$$

The next lemma follows plainly from [21, Theorem 4] and [18, Proposition 1].

Lemma 2.6. *Let $f \in \mathrm{H}_k^*$, $d_j \in \mathbb{N}$, $1 \leqslant m_j \leqslant 4$ and $0 \leqslant n_j \leqslant m_j$ for $1 \leqslant j \leqslant J$. Let $\mathfrak{L}_{\mathbf{m},\mathbf{n}}^{\mathbf{d}}(s)$ be defined as in (2.19). Then for any $\varepsilon > 0$, we have*

$$\int_T^{2T} \big|\mathfrak{L}_{\mathbf{m},\mathbf{n}}^{\mathbf{d}}(\sigma + \mathrm{i}\tau)\big|^2 \mathrm{d}\tau \ll_{f,\varepsilon,\mathbf{d},\mathbf{m},\mathbf{n}} T^{2A(\mathbf{d},\mathbf{m},\mathbf{n})(1-\sigma)+\varepsilon} \tag{2.20}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1$ and $T \geqslant 1$; and

$$\mathfrak{L}_{\mathbf{m},\mathbf{n}}^{\mathbf{d}}(\sigma + \mathrm{i}\tau) \ll_{f,\varepsilon,\mathbf{d},\mathbf{m},\mathbf{n}} (|\tau| + 1)^{\max\{A(\mathbf{d},\mathbf{m},\mathbf{n})(1-\sigma),\, 0\}+\varepsilon} \tag{2.21}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1 + \varepsilon$ and $|\tau| \geqslant 1$.

3. Proof of Theorem 1

By the Perron formula [25, Corollary II.2.1] with (1.3), we can write

$$\sum_{n \leqslant x} \lambda_f(n)^\ell = \frac{1}{2\pi \mathrm{i}} \int_{1+\varepsilon-\mathrm{i}T}^{1+\varepsilon+\mathrm{i}T} F_\ell(s) \frac{x^s}{s} \mathrm{d}s + O_{f,\varepsilon}\bigg(\frac{x^{1+\varepsilon}}{T}\bigg)$$

uniformly for $2 \leqslant T \leqslant x$, where the implied constant depends only on f and ε. In view of Lemma 2.1, the point $s = 1$ is the only possible pole of the integrand in the rectangle $\kappa \leqslant \sigma \leqslant 1 + \varepsilon$ and $|\tau| \leqslant T$ for any $\kappa \in [\frac{1}{2} + \varepsilon, 1)$. The residue at $s = 1$ is equal to $xP_\ell(\log x)$ for $\ell = 4, 6, 8$ and $P_\ell \equiv 0$ if $\ell = 3, 5, 7$. Thus,

$$\sum_{n \leqslant x} \lambda_f(n)^\ell = xP_\ell(\log x) - \frac{1}{2\pi \mathrm{i}} \int_{\mathscr{L}} F_\ell(s) \frac{x^s}{s} \mathrm{d}s + O_{f,\varepsilon}\bigg(\frac{x^{1+\varepsilon}}{T}\bigg),$$

where $\mathscr{L}$ is the contour joining $1 + \varepsilon + \mathrm{i}T$, $\kappa + \mathrm{i}T$, $\kappa - \mathrm{i}T$, $1 + \varepsilon - \mathrm{i}T$ with straight lines. The absolute convergence of $H_j(s)$ for $\Re e s \geqslant \frac{1}{2} + \varepsilon$ yields $H_\ell(s) \ll_{f,\varepsilon} 1$ in the same half plane. Hence the preceding formula can be written as

$$\sum_{n \leqslant x} \lambda_f(n)^\ell = xP_\ell(\log x) + O_{f,\varepsilon}\bigg(\frac{x^{1+\varepsilon}}{T} + \mathfrak{R}_\ell^{\mathrm{h}} + \mathfrak{R}_\ell^{\mathrm{v}}\bigg), \tag{3.1}$$

where

$$\mathfrak{R}_\ell^{\mathrm{h}} := \frac{1}{T} \int_\kappa^{1+\varepsilon} |G_\ell(\sigma + \mathrm{i}T)| x^\sigma \mathrm{d}\sigma,$$

$$\mathfrak{R}_\ell^{\mathrm{v}} := x^\kappa \int_1^T |G_\ell(\kappa + \mathrm{i}\tau)| \frac{\mathrm{d}\tau}{\tau} \ll_{f,\varepsilon} x^{\kappa+\varepsilon} \sup_{1 \leqslant T_1 \leqslant T} \frac{1}{T_1} \int_{T_1}^{2T_1} \big|G_\ell(\kappa + \mathrm{i}\tau)\big| \mathrm{d}\tau.$$

Next we shall treat only the case $\ell = 3$, since the other cases are similar.

According to (2.17) and (2.21), we have

$$\begin{aligned} \mathfrak{R}_3^{\mathrm{h}} &\ll_{f,\varepsilon} \frac{1}{T}\int_\kappa^{1+\varepsilon} T^{\{2(2/3)+(4/2)\}(1-\sigma)+\varepsilon} x^\sigma \mathrm{d}\sigma \\ &\ll_{f,\varepsilon} T^{7/3+\varepsilon}\int_\kappa^{1+\varepsilon}\left(\frac{x}{T^{10/3}}\right)^\sigma \mathrm{d}\sigma \\ &\ll_{f,\varepsilon} \frac{T^{7/3+\varepsilon}}{\log x}\left(\frac{x}{T^{10/3}}\right)^{1+\varepsilon} \\ &\ll_{f,\varepsilon} \frac{x^{1+\varepsilon}}{T} \end{aligned} \tag{3.2}$$

provided $T \leqslant x^{3/10}$.

In order to estimate $\mathfrak{R}_3^{\mathrm{v}}$, we take $\kappa = \frac{5}{8}$ and apply the Cauchy-Schwarz inequality. Consequently,

$$\mathfrak{R}_3^{\mathrm{v}} \ll_f x^{5/8+\varepsilon} \sup_{1\leqslant T_1\leqslant T} I_{3,1}(T_1)^{1/2} I_{3,2}(T_1)^{1/2} T_1^{-1}, \tag{3.3}$$

where

$$I_{3,1}(T_1) := \int_{T_1}^{2T_1} \big|L(\tfrac{5}{8}+\mathrm{i}\tau, f)\big|^4 \mathrm{d}\tau, \quad I_{3,2}(T_1) := \int_{T_1}^{2T_1} \big|L(\tfrac{5}{8}+\mathrm{i}\tau, \mathrm{sym}^3 f)\big|^2 \mathrm{d}\tau.$$

By (2.16) and (2.20), we get

$$I_{3,1}(T_1) \ll_{f,\varepsilon} T_1^{1+\varepsilon} \qquad \text{and} \qquad I_{3,2}(T_1) \ll_{f,\varepsilon} T_1^{4(1-5/8)+\varepsilon}.$$

Inserting into (3.3), it follows that

$$\mathfrak{R}_3^{\mathrm{v}} \ll_{f,\varepsilon} x^{5/8+\varepsilon} T^{1/4+\varepsilon} \tag{3.4}$$

Combining (3.2) and (3.4) with (3.1) and $T = x^{3/10}$, we obtain the required result.

4. Proof of Theorem 2

To facilitate our proof, we give a finer decomposition of $F_\ell(s)$ in (1.8).

Lemma 4.1. *For $\ell = 2, 3, 4, 5, 6$, the Dirichlet series $F_\ell(s)$ admits the factorization*

$$F_\ell(s) = G_\ell(s)\Psi_\ell(2s)\Upsilon_\ell(s) \tag{4.1}$$

where $G_\ell(s)$ is defined as in Lemma 2.1,

$$\Psi_\ell(s) = \prod_{1\leqslant j\leqslant[\ell/2]} G_{2(\ell-2j)}(s)^{-C(\ell,2j)} \times \prod_{1\leqslant j\leqslant[(\ell-1)/2]} G_{2(\ell-1-2j)}(s)^{C(\ell,2j+1)} \tag{4.2}$$

with ($G_0(s) = \zeta(s)$, $G_1(s) = L(s,f)$ and $G_2(s) = \zeta(s)L(s,\mathrm{sym}^2 f)$)

$$C(\ell,d) := \binom{\ell}{d}\big(2^{d-1}-1\big)$$

and $\Upsilon_\ell(s)$ is defined by a Dirichlet series that is absolutely convergent in $\Re\mathrm{e}\, s > \frac{1}{3}$. Besides, the meromorphic function $\Psi_\ell(s)$ has no pole on the line $\Re\mathrm{e}\, s = 1$.

Remarks (i) In view of (1.5), we have $\Psi_2(s) = \zeta(s)^{-1}$ and $\Upsilon_2(s) \equiv 1$.

(ii) The factorization (4.1) holds for all $\ell \in \mathbb{N}$ (with $G_\ell(s)$ being defined as $F_\ell(s)$ in [13, Lemma 7.1]). We confine $\ell \leqslant 6$ for unconditional results.

Proof. It suffices to compare the local factors on both sides of (4.1), and check that $\log F_\ell(s)$ and $\log(G_\ell(s)\Psi_\ell(2s))$ coincide up to p^{-2s} for suitable exponents $C(\ell,d)$.

Write $\lambda_f(p)=2\cos\theta$, then $\lambda_f(p^\nu)=T_\nu(2\cos\theta)=U_\nu(\cos\theta)$. The p-local factors of $F_\ell(s)$ and its logarithm $\log F_\ell(s)$ are respectively

$$1+\sum_{\nu\geqslant 1}\frac{U_\nu(\cos\theta)^\ell}{p^{\nu s}} \quad \text{and} \quad \frac{U_1(\cos\theta)^\ell}{p^s}+\frac{U_2(\cos\theta)^\ell-\frac12 U_1(\cos\theta)^{2\ell}}{p^{2s}}+O\Big(\frac{1}{p^{3s}}\Big).$$

Recalling that (2.1) follows from (2.2) and the fact that $U_1(x)^\ell=(2x)^\ell$, the local factor of $\log G_\ell(s)$ is

$$\sum_{\nu\geqslant 1}\frac{U_1(\cos(\nu\theta))^\ell}{\nu}p^{-\nu s}. \tag{4.3}$$

Hence, the difference between the local factors of $\log F_\ell(s)$ and $\log G_\ell(s)$ equals

$$\big(U_2(\cos\theta)^\ell-\tfrac12 U_1(\cos\theta)^{2\ell}-\tfrac12 U_1(\cos(2\theta))^\ell\big)p^{-2s}+O\big(p^{-3s}\big). \tag{4.4}$$

Observing that $U_2=U_1^2-1$ and $U_1(\cos(2\theta))=U_1(\cos\theta)^2-2$, the coefficient of p^{-2s} in (4.4) equals

$$\begin{aligned}&\sum_{d=2}^{\ell}(-1)^d(1-2^{d-1})\binom{\ell}{d}U_1(\cos\theta)^{2(\ell-d)}\\ &=\sum_{j=1}^{[(\ell-1)/2]}C(\ell,2j+1)U_1(\cos\theta)^{2(\ell-1-2j)}-\sum_{j=1}^{[\ell/2]}C(\ell,2j)U_1(\cos\theta)^{2(\ell-2j)}.\end{aligned}$$

In view of (4.3), we can replace the first term in (4.4) by the local factors of

$$\sum_{j=1}^{[(\ell-1)/2]}C(\ell,2j+1)\log G_{2(\ell-1-2j)}(2s)-\sum_{j=1}^{[\ell/2]}C(\ell,2j)\log G_{2(\ell-2j)}(2s) \tag{4.5}$$

up to $O(p^{-3s})$. This verifies the factorization of $F_\ell(s)$.

It remains to evaluate the order of $\Psi_\ell(s)$ at $s=1$. $G_{2j}(s)$ has a pole of order $g_{2j}=(2j)!/(j!(j+1)!)$ at $s=1$, i.e. $g_0=1, g_2=1, g_4=2, g_6=5, g_8=14$, and the values of $C(\ell,d)$ $(2\leqslant d\leqslant \ell\leqslant 6)$ are given in the table:

ℓ, d	2	3	4	5	6
2	*1*				
3	*3*	*3*			
4	*6*	*12*	*7*		
5	*10*	*30*	*35*	*15*	
6	*15*	*60*	*105*	*90*	*31*

Hence the order of $\Psi_\ell(s)$ at $s=1$ (which is negative for a pole) is given by (4.5) with $\log G_{2r}(2s)$ replaced by $-g_{2r}$, and is equal to $1,0,7,10,61$ for $\ell=2,3,4,5,6$ respectively. This completes the proof. $\square$

We are ready to prove Theorem 2. In light of Lemma 4.1, we write

$$F_\ell(s)=\frac{f_1(s)}{\prod_{1\leqslant j\leqslant \ell/2}g_j(2s)}h(s)$$

where $f_1(s) = G_\ell(s)$, $g_j(2s) = G_{2(\ell-2j)}(2s)^{C(\ell,2j)}$ and

$$h(s) = \prod_{j=1}^{[(\ell-1)/2]} G_{2(\ell-1-2j)}(2s)^{C(\ell,2j+1)}\Upsilon_\ell(s).$$

The conditions (A)-(E) required in [12, Theorem 2] are verified with the following choice of parameters (in the notation of [12]):

$$\begin{cases} J=[\ell/2], \quad n_j=2, \quad \sigma_*^j = 1-2^{-\ell}-10^{-\ell} \quad (1\leqslant j\leqslant J),\\ K=1, \quad m_1=1, \quad \kappa_1=2^\ell, \quad \sigma_1^*=0,\\ \alpha = 2^{-1}(1-2^{-\ell})>1/3 \quad (\text{as } \ell\geqslant 2).\end{cases} \tag{4.6}$$

Apparently $f_1(s), g_j(s)$ and $h(s)$ are absolutely convergent Dirichlet series for $\Re es > 1$:

$$f_1(s) = \sum_{n\geqslant 1} a_1(n)n^{-s}, \qquad g_j(s) = \sum_{n\geqslant 1} b_j(n)n^{-s} \qquad h(s) = \sum_{n\geqslant 1} c(n)n^{-s}$$

with $a_1(1) = b_j(1) = c(1) = 1$ and $a_1(n), b_j(n), b_j^*(n), c(n) \ll_\varepsilon n^\varepsilon$ for any $\varepsilon > 0$ and all $n \geqslant 1$, thanking to the Deligne inequality (1.3). Note that $b_j^*(n)$ is the inverse arithmetic function of $b_j(n)$ with respect to Dirichlet convolution. Conditions (A), (B) and (D) in [12] are quite obviously valid, for instance,

$$\left|\frac{f_1(\sigma+\mathrm{i}\tau)}{f_1(1-\sigma+\mathrm{i}\tau)}\right| \gg |\tau|^{2^\ell(1/2-\sigma)}$$

for $\sigma = \alpha$ and $|\tau| \gg 1$, as the degree of $G_\ell(s)$ is 2^ℓ.

The crucial condition (C) concerns the zero density of $g_j(s)$. Denote by $N_L(\sigma_0, T)$ the number of zeros of a generic L-function $L(s)$ in $\sigma \geqslant \sigma_0$ and $0 \leqslant \tau \leqslant T$. Condition (C) will hold if $N_{g_j}(\sigma, T) \ll T^{1-1/10}$ when $\sigma = \sigma_*^{(j)} = 2\alpha - 10^{-\ell}$. To this end, we invoke [20, Theorem 1]: if $L(s)$ is in the Selberg class and of degree d, then

$$N_L(\sigma, T) \ll T^{d(1-\sigma)+\varepsilon} \qquad \text{for } 2/d \leqslant \sigma < 1.$$

Each factor $L(s)$ in $G_{2(\ell-2j)}(s)$ is belonged to Selberg class and has degree $d \leqslant (\ell-2j+1)^2$. If $3 \leqslant d \leqslant (\ell-2j+1)^2 \leqslant (\ell-1)^2$, then $2\alpha - 10^{-\ell} \geqslant 2/3 \geqslant 2/d$ and so $N_L(\sigma, T) \ll T^{A+\varepsilon}$ for $\sigma = 2\alpha - 10^{-\ell}$, where

$$A \leqslant d\big(1-2\alpha+10^{-\ell}\big) = d\big(2^{-\ell}+10^{-\ell}\big) \leqslant (\ell-1)^2\big(2^{-\ell}+10^{-\ell}\big) < \tfrac{4}{5}.$$

When $d = 1$ or 2, $L(s) = \zeta(s)$ or $L(s, f)$ and thus $N_L(\sigma, T) \ll T^{0.9}$ for $\sigma = 2\alpha - 10^{-\ell}$. (This estimate is crude but sufficient.) Condition (C) is hence satisfied. Condition (E) is also valid under our choice of parameters in (4.6).

As Theorems 1 and 2 in [12] are applicable, our proof of Theorem 2 is complete.

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